Lagrange Formalism & Gauge Theories

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Schedule for today

• How do I know that the gauge field should be a boson?

• What is the defining characteristic of a Lie group?

1. Lagrange formalism
2. Local gauge transformations
3. Lie-groups & (Non-)Abelian transformations
Lagrange formalism & gauge transformations

Joseph-Louis Lagrange
(*25. January 1736, † 10. April 1813)
Lagrange formalism (classical field theories)

- All information of a physical system is contained in the action integral:

  \[ S = \int \mathcal{L}(\partial_{\mu} \phi, \phi) dx_{\mu} \]

  Lagrange Density: \( \mathcal{L} \) (Generalization of canonical coordinates)

  Field: \( \phi \)

- Equations of motion can be derived from the Euler-Lagrange formalism:

  \[ \partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \]

  (From variation of action)

- What is the dimension of \( \mathcal{L} \)?
Lagrange formalism (classical field theories)

- All information of a physical system is contained in the *action integral*:

\[
S = \int \mathcal{L}(\partial_\mu \phi, \phi) dx_\mu
\]

Field: \( \phi \) (Generalization of canonical coordinates)

- Equations of motion can be derived from the *Euler-Lagrange formalism*:

\[
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0
\]

(From variation of action)

- What is the dimension of \( \mathcal{L} \)?

\([\mathcal{L}] = \text{GeV}^4\)
Lagrange density for (free) bosons & fermions

For Bosons:

\[ \mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* \]

For Fermions:

\[ \mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \]

- Proof by applying *Euler-Lagrange formalism* (shown only for Bosons here):

\[ \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^* )} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0 \]

\[ \partial^\mu \partial_\mu \phi - m^2 \phi \rightarrow (\partial^\mu \partial_\mu + m^2) \phi = 0 \]

- NB:
  - There is a distinction between \( \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi) } \) and \( \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*) } \).
  - Most trivial is variation by \( \bar{\psi} \), least trivial is variation by \( \psi \).
Global phase transformations

• The Lagrangian density is **covariant under global phase transformations** (shown here for the fermion case only):

\[
\begin{align*}
\psi(\vec{x}, t) & \to \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t) \\
\bar{\psi}(\vec{x}, t) & \to \psi'(\vec{x}, t) = \bar{\psi}(\vec{x}, t) e^{-i\vartheta}
\end{align*}
\]

(Global phase transformation) \( \vartheta \neq \vartheta(\vec{x}, t) \)

\[
\mathcal{L}' = \bar{\psi}' (i\gamma^\mu \partial_\mu - m) \psi' = \bar{\psi} e^{-i\vartheta} (i\gamma^\mu \partial_\mu - m) e^{i\vartheta} \psi
\]

\[
= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \mathcal{L}
\]

• Here the phase \( \vartheta \) is **fixed at each point in space** \( \vec{x} \) **at any time** \( t \).

• What happens if we allow different phases at each point in \( (\vec{x}, t) \)?
Local phase transformations

- This is not true for local phase transformations:

\[
\begin{align*}
\psi(x, t) &\rightarrow \psi'(x, t) = e^{i\vartheta} \psi(x, t) \\
\bar{\psi}(x, t) &\rightarrow \psi'(x, t) = \bar{\psi}(x, t)e^{-i\vartheta}
\end{align*}
\]

(Local phase transformation)

\[
\vartheta = \vartheta(x, t)
\]

\[
\mathcal{L}' = \bar{\psi}' (i\gamma^{\mu} \partial_{\mu} - m) \psi' = \bar{\psi} e^{-i\vartheta} (i\gamma^{\mu} \partial_{\mu} - m) e^{i\vartheta} \psi
\]

\[
= \bar{\psi} (i\gamma^{\mu} (\partial_{\mu} + i\partial_{\mu} \vartheta) - m) \psi \neq \mathcal{L}
\]

Connects neighboring points in \((x, t)\)

Breaks invariance due to \(\partial_{\mu}\) in \(\mathcal{L}\).

\[
\psi(x+\Delta x) - \psi(x) \over \Delta x
\]
The covariant derivative

- Covariance can be enforced by the introduction of the **covariant derivative**:
\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu \]
with the corresponding transformation behavior

\[
\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t) \\
\bar{\psi}(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = \bar{\psi}(\vec{x}, t)e^{-i\vartheta} \\
D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu \vartheta
\]

\( \text{(Local phase transformation)} \)

\( \vartheta = \vartheta(\vec{x}, t) \)

\( \text{(Arbitrary gauge field)} \)

\[ L' = \bar{\psi}' (i\gamma^\mu D'_\mu - m) \psi' = \bar{\psi}e^{-i\vartheta} (i\gamma^\mu (D_\mu - i\partial_\mu \vartheta) - m) e^{i\vartheta} \psi \]

\[ = \bar{\psi} (i\gamma^\mu (D_\mu - i\partial_\mu \vartheta + i\partial_\mu \vartheta) - m) \psi = L \]

- **NB**: What is the transformation behavior of the gauge field \( A_\mu \)?
The covariant derivative

- Covariance can be enforced by the introduction of the **covariant derivative**:
  \[ \partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu \]
  with the corresponding transformation behavior

\[ \psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t) \]
\[ \overline{\psi}(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = \overline{\psi}(\vec{x}, t)e^{-i\vartheta} \]
\[ D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu \vartheta \]

(Local phase transformation)

(Arbitrary gauge field)

\[ \mathcal{L}' = \overline{\psi}' \left( i\gamma^\mu D'_\mu - m \right) \psi' = \overline{\psi}e^{-i\vartheta} \left( i\gamma^\mu (D_\mu - i\partial_\mu \vartheta) - m \right) e^{i\vartheta} \psi \]
\[ = \overline{\psi} \left( i\gamma^\mu (D_\mu - i\partial_\mu \vartheta + i\partial_\mu \vartheta) - m \right) \psi = \mathcal{L} \]

**NB:** What is the transformation behavior of the gauge field \( A_\mu \)?

\[ A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \vartheta \]

known from electro-dynamics!
The gauge field

- Possible to allow arbitrary phase $\vartheta$ of $\psi(\vec{x}, t)$ at each individual point in $(\vec{x}, t)$.
- Requires introduction of a mediating field $A_\mu$, which transports this information from point to point.

\[ \psi(\vec{x}, t) \xrightarrow{e} A_\mu \xrightarrow{e} \psi(\vec{x}', t') \]

- The gauge field $A_\mu$ couples to a quantity $e$ of the external field $\psi(\vec{x}, t)$, which can be identified as the electric charge.
- The gauge field $A_\mu$ can be identified with the photon field.
The interacting fermion

- The introduction of the covariant derivative leads to the \textit{Lagrangian density} of an interacting fermion with electric charge $e$:

$$\mathcal{L}_{IA} = \bar{\psi} \left( i \gamma^\mu (D_\mu - m) \right) \psi$$

$$= \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi - e \bar{\psi} \gamma^\mu A_\mu \psi$$

- Description still misses dynamic term for a free gauge boson field (=photon).
Gauge field dynamics

- Ansatz:

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ \mathcal{L}_{\text{kin}} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} \]

*(Field-Strength tensor) (Free photon field)*

- Motivation:

- Variation of the action integral

\[ S = \delta \int (-m ds - e A_\mu dx^{\mu}) \]

in classical field theory, leads to

\[ m \frac{d\nu_\mu}{ds} = e (\partial_\mu A_\nu - \partial_\nu A_\mu) \nu^{\nu} \]

- Can also be obtained from:

\[ F_{\mu \nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{i}{\epsilon} [D_\mu, D_\nu] \]

- \( F_{\mu \nu} F^{\mu \nu} \) is manifest Lorentz invariant.

- \( A_\mu \) appears quadratically \( \rightarrow \) linear appearance in variation that leads to equations of motion \( \rightarrow \) superposition of fields.

- Check that \( F_{\mu \nu} \) is gauge invariant.
Complete Lagrangian density

- Application of $U(1)$ gauge symmetry leads to Lagrangian density of QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i\gamma^\mu (D_\mu - m)) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e\bar{\psi} \gamma^\mu A_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Free Fermion Field \quad IA Term \quad Gauge

(Interacting Fermion)

- Variation of $\bar{\psi}$:

$$i\gamma^\mu (\partial_\mu - m) \psi + e\gamma^\mu A_\mu \psi = 0$$

- Derive equations of motion for an interacting boson.
Complete Lagrangian density

- Application of $U(1)$ gauge symmetry leads to Largangian density of QED:

\[
\mathcal{L}_{\text{QED}} = \bar{\psi} (i\gamma^\mu (D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e\bar{\psi} \gamma^\mu A_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

Free Fermion Field \hspace{1cm} IA Term \hspace{1cm} Gauge

*(Interacting Fermion)*

- Variation of $A_\mu$:

\[
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu} = 0
\]

\[
\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial_\mu \partial^\mu A_\mu - \partial^\nu \partial_\mu A^\mu) = 0
\]

\[
(\partial_\mu \partial^\mu - 0) A_\mu = 0
\]

*(Klein-Gordon equation for a massless particle)*
Summary (Abelian) gauge field theories

\[ \psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t) \]
\[ \overline{\psi}(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = \overline{\psi}(\vec{x}, t) e^{-i\vartheta} \]

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu \]
\[ D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu \vartheta \]
\[ A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \vartheta \]

\[ F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \]
\[ F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu} \]

\[ \mathcal{L} = \overline{\psi} \left( i\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

(Local gauge invariance)

(Covariant derivative)

(Field strength tensor)

(Lagrange density)
Review of Lie-Groups

Marius Sophus Lie
(*17. December 1842, † 18. February 1899)
$\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t)$

$U(1)$ phase transformation.

- $U(1)$ is a group of unitary transformations in $\mathbb{R}^n$ with the following properties:

  $G \in U(n)$  \hspace{1cm} $G^\dagger G = \mathbb{I}_n$  \hspace{1cm} $|\text{det} \ G| = 1$

- Splitting an additional phase from $G$ one can reach that $\text{det} \ G = 1$:

  $U(n) = U(1) \times SU(n)$

\[|\text{det} \ G| = 1 \quad \text{(Unitary transformations)} \quad \quad \quad \text{det} \ G = +1 \quad \text{(Special unitary transformations)}\]
Infinitesimal $\rightarrow$ finite transformations

- The $SU(n)$ can be composed from infinitesimal transformations with a continuous parameter $\vartheta \in \mathbb{R}$:

$$G_{\text{finite}} = I_n + i\vartheta_{\text{finite}}t \quad (\vartheta_{\text{finite}} \in \mathbb{R}, \, t \in \mathcal{M}(n \times n))$$

- The set of $G$ forms a Lie-Group.

- The set of $t$ forms the tangential-space or Lie-Algebra.
Properties of $t$

- **Hermitian:**
  \[ G^\dagger G = \mathbb{I}_n \]
  \[ = (\mathbb{I}_n - i\vartheta t^\dagger) (\mathbb{I}_n + i\vartheta t) = \mathbb{I}_n + i\vartheta (t - t^\dagger) + O(\vartheta^2) \]
  \[ t = t^\dagger \]

- **Traceless** (example $SU(n)$):
  \[ \det G = \det (\mathbb{I}_n + i\vartheta t) \]
  \[ = 1 + i\vartheta \text{Tr}(t) + O(\vartheta^2) \]
  \[ \text{Tr}(t) = 0 \]

- **Dimension of tangential space:**
  \[
  \begin{pmatrix}
  * & * & * & * & * \\
  * & * & * & * & * \\
  * & * & * & * & * \\
  * & * & * & * & * \\
  * & * & * & * & *
  \end{pmatrix}
  \]
  - $n$ real entries in diagonal.
  - $\frac{1}{2} \cdot n(n - 1)$ complex entries in off-diagonal.
  - $-1$ for $SU(n)$ for det req.

\[ \begin{cases} 
  \text{\(U(n)\) has } n^2 \text{ generators.} \\
  \text{\(SU(n)\) has } (n^2 - 1) \text{ generators.}
\end{cases} \]
Examples that appear in the SM ($U(1)$)

- $U(1)$ transformations (equivalent to $O(2)$):
  - Number of generators: $1^2 = 1$  
  
  **NB:** what is the Generator?
Examples that appear in the SM ($U(1)$)

- $U(1)$ transformations (equivalent to $O(2)$):
  - Number of generators: $1^2 = 1$  
    **NB:** what is the Generator?  
    ➡️ The generator is 1.
Examples that appear in the SM ($SU(2)$)

- $SU(2)$ transformations (equivalent to $O(3)$):
  - Number of generators: $\left(2^2 - 1\right) = 3$
  - i.e. there are 3 matrices $\{t_j\}$, which form a basis of traceless hermitian matrices, for which the following relation holds:
    \[ G = e^{i\sum_{j=1}^{3} \varphi_j t_j} \]
  - Explicit representation:
    \[
    t_j = \frac{1}{2} \sigma_j \quad (j = 1 \ldots 3)
    \]
    (3 Pauli matrices)
    \[
    [t_i, t_j] = i\epsilon_{ijk} t_k
    \]
    - algebra closes.
    - structure constants of $SU(2)$. 

Examples that appear in the SM ($SU(3)$)

- $SU(3)$ transformations (equivalent to $O(4)$):
  
  - Number of generators: $(3^2 - 1) = 8$
  
  - i.e. there are 8 matrices $\{T_j\}$, which form a basis of traceless hermitian matrices, for which the following relation holds:
    
    $$G = e^{i \sum_{j=1}^{8} \vartheta_j T_j}$$
  
  - Explicit representation:
    
    $T_j = \frac{1}{2} \lambda_j \quad (j = 1 \ldots 8)$  

    (8 Gell-Mann matrices)

    $$[T_i, T_j] = i f_{ijk} T_k$$

    - algebra closes.
    
    - structure constants of $SU(3)$. 
(Non-)Abelian symmetry transformations

- Example $O(3)$ ($90^\circ$ rotations in $\mathbb{R}^3$):

**Diagram:**

- Switch $z$ and $y$: 

  - $x$ axis
  - $y$ axis
  - $z$ axis
  - $x$ axis
  - $z$ axis
  - $y$ axis
(Non-)Abelian symmetry transformations

- Example $O(3)$ ($90^\circ$ rotations in $\mathbb{R}^3$):

  switch $z$ and $y$: 

  - $x$ rotation
  - $y$ rotation
  - $z$ rotation

  ![Diagram](image)
(Non-)Abelian symmetry transformations

- Example $O(3)$ ($90^\circ$ rotations in $\mathbb{R}^3$):

  switch $z$ and $y$:

  cyclic permutation:
(Non-)Abelian symmetry transformations

- Example $O(3)$ ($90^\circ$ rotations in $\mathbb{R}^3$):

switch $z$ and $y$:

cyclic permutation:
Abelian vs. Non-Abelian gauge theories

\[ \psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta} \psi(\vec{x}, t) \]
\[ \bar{\psi}(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = \bar{\psi}(\vec{x}, t)e^{-i\vartheta} \]

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu \]
\[ D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu \vartheta \]
\[ A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \vartheta \]

\[ F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \]
\[ F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu} \]

\[ \mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

\[ \psi(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = e^{i\vartheta_a t_a} \psi(\vec{x}, t) \]
\[ \bar{\psi}(\vec{x}, t) \rightarrow \psi'(\vec{x}, t) = \bar{\psi}(\vec{x}, t)e^{-i\vartheta_a t_a} \]

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu + igW_{\mu,a} t_a \]
\[ D_\mu \rightarrow D'_\mu = D_\mu + i[\vartheta_a t_a, D_\mu] \]
\[ W_\mu \rightarrow W'_\mu = W_\mu + i[\vartheta_a t_a, W_\mu t_a] \]
\[ -\frac{1}{g} \partial_\mu (\vartheta_a t_a) \]
\[ W_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu \]
\[ + ig[W_\mu, W_\nu] \]
\[ W_{\mu\nu} \rightarrow W'_{\mu\nu} = W_{\mu\nu} + i[\vartheta_a t_a, W_{\mu\nu}] \]

\[ \mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} W_{a\mu\nu} W^{a\mu\nu} \]
Concluding remarks

- Reprise of Lagrange formalism.
- Requirement of local gauge symmetry leads to coupling structure of QED.
- Extension to more complex symmetry operations will reveal non-trivial and unique coupling structure of the SM and thus describe all known fundamental interactions.
- Next lecture on layout of the electroweak sector of the SM, from the non-trivial phenomenology to the theory.
- Prepare “The Higgs Boson Discovery at the Large Hadron Collider” Section 2.2.